Structural vibration
Learning summary

By the end of this chapter you should have learnt about:

- Natural frequencies and mode shapes
- Response of damped single-degree-of-freedom systems
- Response of damped multi-degree-of-freedom systems
- Experimental modal analysis
- Approximate methods
- Vibration control techniques.

6.2 Natural frequencies and mode shapes – key points

By the end of this section you should have learnt:

- a systematic approach for setting up the equations of motion for single-degree-of-freedom and lumped mass–spring systems using the following three steps:

  - Step 1 convert the physical structure into a dynamic mass–spring model

  - Step 2 draw free-body diagram(s)

  - Step 3 apply the appropriate form(s) of Newton's second law of motion to give the equation(s) of motion for the system

6.2 Natural frequencies and mode shapes – key points

- how to obtain the natural frequency of single-degree-of-freedom systems from the coefficients in the equation of motion
- the equations of motion for lumped mass–spring systems and how to obtain the natural frequencies and the corresponding mode shapes
- how to set up the boundary condition equations for shaft/beam vibration problems and obtain the frequency equation and an expression for the mode shapes.

6.3 Response of damped single-degree-of-freedom systems – key points

By the end of this section you should have learnt:

- how to obtain the response of single-degree-of-freedom mass–spring-damper systems for the cases of ‘free’ vibration and for harmonic (sinusoidal) and arbitrary periodic excitation.

6.4 Response of damped multi-degree-of-freedom systems – key points

By the end of this section you should have learnt:

- how to uncouple the equations of motion of a proportionally damped multi-degree-of-freedom structure by making use of the orthogonality properties of the modal matrix
- how to formulate response expressions, including the frequency response function.

6.5 Experimental modal analysis – key points

By the end of this section you should have learnt:

- how the expression for the frequency response function for a multi-degree-of-freedom structure can be used to identify the natural frequencies, damping and mode shapes of structures from experimental tests.

6.6 Approximate methods – key points

By the end of this section you should have learnt:

- how to use Dunkerley's method and Rayleigh's method to obtain estimates of the lowest natural frequency of structures
- how to create a single-degree-of-freedom approximation of a more complex structure and use it to estimate the structure's response.

6.7 Vibration control techniques – key points

By the end of this section you should have learnt:

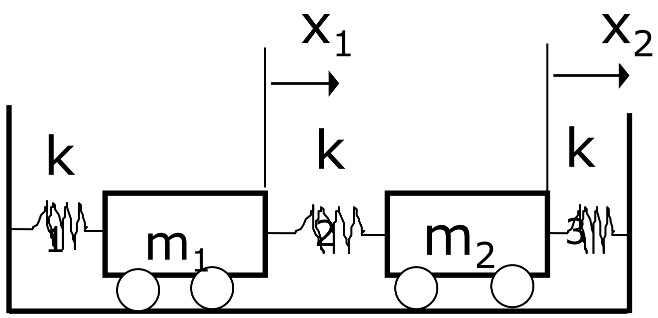
- how to derive the transmissibility expression that gives the ratio of transmitted force to applied force (or of transmitted displacement to applied displacement) as a function of excitation frequency
- how to select suitable isolators to achieve a given isolation efficiency
- how tuned vibration absorbers can suppress resonant vibration in situations where sinusoidal excitation coincides with a natural frequency of the structure.

Multi-degree of Freedom Systems

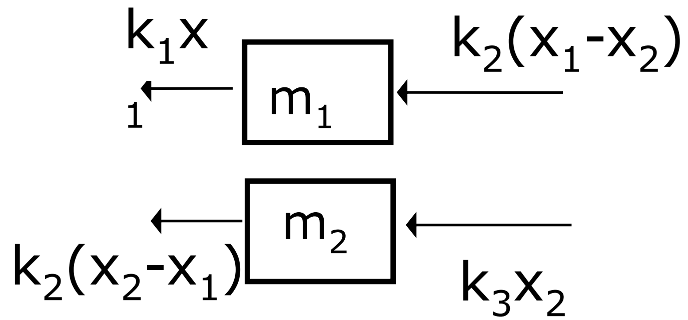
- Motivation: Many systems are too complex to be represented by a single degree of freedom model.
- Objective of this chapter: Understand vibration of systems with more than one degree of freedom.

Free-vibration of undamped two-degree of freedom system

- We learn how to analyze free vibration by considering an example



Deriving equations of motion



$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = 0$$

Interpretation of coefficients

First equation: k_1+k_2 is the force on m_1 needed to move it slowly by one unit while m_2 is held stationary. –k_2 is the force on m_1 need to hold it steady if m_2 is displaced slowly by one unit.

Second equation: k_1+k_2 is the force on m_2 needed to move it slowly by one unit while m_1 is held stationary. –k_2 is the force on m_2 need to hold is steady if m_1 is displaced slowly by one unit.

Maxwell reciprocity theorem: the force on m_1 need to hold it steady if m_2 is displaced slowly by one unit=force on m_2 need to hold is steady if m_1 is displaced slowly by one unit.

Special case

- Let m1=m2=m and k1=k2=k3=k. Then:

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0$$

Solution of equations of motion

We know from experience that:

$$x_1(t) = A\sin(\omega t)$$
$$x_2(t) = B\sin(\omega t)$$

Substituting the above equation to eq. of motion, we obtain two eqs w.r.t. A, B :

$$(-mA\omega^2 + 2kA - kB)\sin\omega t = 0$$
$$(-mB\omega^2 + 2kB - kA)\sin\omega t = 0$$

The above two equations are satisfied for every t

$$-mA\omega^2 + 2kA - kB = 0$$
$$-mB\omega^2 + 2kB - kA = 0$$

Trivial solution A, B=0. In order to have a nontrivial solution:

$$\det\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} = 0$$

$$\omega = \pm\sqrt{\frac{k}{m}}, \pm\sqrt{\frac{3k}{m}}$$

First natural frequency:

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow A = B$$

The displacement for the first natural frequency is: $\begin{pmatrix} A \\ A \end{pmatrix}$

This vector is called mode shape. Constant cannot be deter
Usually assume first entry is 1. Therefore, mode shape is: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus, both masses move in phase and the have the same amplitudes.

Second natural frequency:

$$\omega = \sqrt{\frac{3k}{m}} \Rightarrow A = -B$$

Mode shape for above frequency: $\begin{pmatrix} A \\ -A \end{pmatrix}$

Usually assume first entry is 1. Therefore, mode shape is:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The two masses move in opposite directions.

Displacements of two masses are sums of displacements in the two modes:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\sqrt{\frac{k}{m}} t + \psi_1) + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(\sqrt{\frac{3k}{m}} t + \psi_2)$$

General expression for vibration of the two-degree of freedom system

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \sin(\omega_1 t + \psi_1) + c_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \sin(\omega_2 t + \psi_2)$$

Observations:

- ~~System motion is superposition of two harmonic (sinusoidal)~~ motions with frequencies ω_1 and ω_2.
- Participation of each mode depends on initial conditions.
- Four unknowns (c_1, c_2, ψ_1, ψ_2) can be found using four initial conditions.
- Can find specific initial conditions so that only one mode is excited.

How to solve a free vibration problem involving a two degree of freedom system

1) Write equations of motion for free vibration (no external force or moment)
2) Assume displacements are sinusoidal waves, and plug in equations of motion: Obtain equation:

$$[C(\omega)]*\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where $$[C(\omega)] = [K] - \omega^2 [M]$$

3) Solve det[C(ω)]=0, obtain two natural frequencies, ω_1, ω_2.
4) Solve,

$$[\mathrm{C}(\omega_1)]*\begin{Bmatrix} \mathrm{A} \\ \mathrm{B} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assume A=1, and obtain first mode shape. Obtain second mode shape in a similar manner.

Free vibration response is:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \sin(\omega_1 t + \psi_1) + c_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \sin(\omega_2 t + \psi_2)$$

5) Find constants from initial conditions.